Accepted for publication in Astronomy Letters, v. 27, N 12, 2001.

Nodal and Periastron Precession of Inclined Orbits in the Field of a Rotating Black Hole .

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Abstract

The inclination of low-eccentricity orbits is shown to significantly affect the orbital parameters, in particular, the Keplerian, nodal precession, and periastron rotation frequencies, which are interpreted in terms of observable quantities. For the nodal precession and periastron rotation frequencies of low-eccentricity orbits in a Kerr field, we derive a Taylor expansion in terms of the Kerr parameter at arbitrary orbital inclinations to the black-hole spin axis and at arbitrary radial coordinates. The particle radius, energy, and angular momentum in the marginally stable circular orbits are calculated as functions of the Kerr parameter j and parameter s in the form of Taylor expansions in terms of j to within $O[j^6]$. By analyzing our numerical results, we give compact approximation formulas for the nodal precession frequency of the marginally stable circular orbits at various s in the entire range of variation of Kerr parameter.

Key words: black holes, precession, disk accretion

Introduction

The X-ray quasi-periodic oscillations (QPOs) discovered in low-mass X-ray binaries (LMXBs) commonly show a variety of modes. In particular, there are horizontal-branch oscillations at low frequencies $\nu_{\rm HBO}$ 1–100 Hz and two peaks at frequencies ν_1 and ν_2 ($\sim$1 kHz) in the power spectrum (van der Klis 2000). In many Z-type sources, as well as in several atoll sources, the frequencies $\nu_{\rm HBO}$ exhibit an almost quadratic dependence on ν_2 (Stella and Vietri 1998; Psaltis *et al.* 1999). For several sources (e.g., for Sco X-1, 4U 1608–52, 4U 1702–43, 4U 1735–44, XTE J2123–058), the frequency difference $\nu_2 - \nu_1 \equiv \Delta\nu$ decreases with increasing ν_2. At the same time, for many Z-type (GX 5-1,GX 17+2, Cyg X-2, GX 340+0, GX 349+2) and atoll (4Γ 0614+09, 4Γ 1636-52, 4Γ 1705-44, Aq1 X-1) sources, the difference $\Delta\nu$ is constant, which allows it to be associated with the stellar rotation frequency (van der Klis 2000). The frequency ν_2 is currently identified with the Keplerian rotation frequency of clumps of matter near a compact object ($\nu_2 \equiv \nu_{\rm K}$) in almost all interpretations.

Cui *et al.* (1998) pointed out that the nodal precession of circular orbits[1] slightly inclined to the equatorial plane of a black hole could be of importance in interpreting the stable QPOs: for several black-hole candidates and microquasars, they found the predicted frequencies of nodal precession to agree (if the Kerr parameter is chosen in the range $0.37 - 0.9$) with the observed QPO frequencies.

Stella and Vietri (1998) justified the formula for the nodal precession frequency $\nu_{\rm nod}$ of orbits slightly inclined to the equatorial plane and identified its even harmonics with the frequency $\nu_{\rm HBO}$. The frequency ν_1 was associated with the periastron rotation frequency $\nu_{\rm per} = \nu_{\rm K} - \nu_r$ of a low-eccentricity orbit (a general-relativity effect). They showed that the theoretical dependence $\nu_r(\nu_\phi)$ for an appropriately chosen mass of the object in the range $1.8 - 2.2 M_\odot$ was in good agreement with the experimental points in the $(\nu_2, \Delta\nu)$ plane for various sources with evolving frequencies. In the Kerr solution, the frequency ν_r becomes zero at the marginally stable orbit. Therefore, when a radiating clump of matter passes from one Keplerian orbit to another[2] by increasing its rotation frequency, the frequency difference between the peaks $\Delta\nu = \nu_r$ decreases and even approaches zero at the marginally stable orbit. This tendency was observed for some of the sources (see above).

Morsink and Stella (1999), Stella and Vietri (1998), and Psaltis *et al.* (1999) showed the theoretical dependences $\nu_{\rm nod}(\nu_{\rm K})$ to agree with the measured source frequencies in the $\nu_2, \nu_{\rm HBO}$ plane if $\nu_{\rm HBO}$ was identified with an appropriate even harmonic of the nodal frequency.

The proper rotation of the source (a neutron star or a black hole) remains a free parameter, and it can be chosen by using the stable frequency observed in LMXBs in the X-ray band during outbursts.

Another free parameter is the orbital inclination s to the equatorial plane of a neutron star or a black hole, which significantly affects the observed nodal precession, periastron rotation, and Keplerian frequencies near the marginally stable orbit. Below, we give an example based on our results. If the mass of a black hole or a neutron star is M, then for the Keplerian frequency in the marginally stable orbit $\nu_\phi = 1.2\ (2.2 M_\odot/M)$ kHz, the frequency of nodal precession $\nu_{\rm nod}$ is, respectively, 123 $(2.2M_\odot/M)$ Hz at $s = 0$; 93.6 $(2.2M_\odot/M)$ Hz at $s = 10°$; 65.3 $(2.2M_\odot/M)$ Hz at $s = 30°$; 52.6 $(2.2M_\odot/M)$ Hz at $s = 45°$; 41.18 $(2.2M_\odot/M)$ Hz at $s = 80°$; and 41.11 $(2.2M_\odot/M)$ Hz at $s = 90°$. The nodal precession frequency changes by almost a factor of 3 as the inclination changes from 90° to 0! Thus, by simultaneously measuring three quantities, M, $\nu_{\rm HBO}$, and ν_2, we can determine the inclination of the marginally stable orbit to the spin axis!

In general, the tilt of an accretion disk to the equatorial plane of a compact object can be finite. The Bardeen–Petterson (1975) hypothesis of the accretion-disk transition into the equatorial plane breaks down when the radiative forces that twist the inner edge of the disk are taken into account. Pringle (1996) showed that a flat disk was unstable to disturbances in the presence of a central radiation source.

Below, we derive formulas for the Keplerian, nodal precession, and periastron rotation frequencies of low-eccentricity orbits at an arbitrary (finite) orbital inclination to the equatorial plane in the form of Taylor expansions in terms of the Kerr parameter. In contrast to previous studies, in which calculations were performed for selected values of the constant Q [with an unclear physical meaning, as was noted by de Felice (1980), and coinciding with the square of the total angular momentum only in the weak-field approximation Q), we consistently

[1]Lense–Thirring (1918) effect.

[2]Through turbulent viscosity and radiative deceleration.

use the smallest latitudinal angle $\theta_- = s$ reached on a bounded trajectory[3] as a trajectory parameter. At $s = \pi/2$, the derived formulas (51) transform into Taylor expansions of the formulas by Okazaki *et al.* (1987). An analysis of our numerical results has yielded compact approximation formulas for the nodal precession frequency of the marginally stable circular orbits in the entire range of Kerr parameters for several finite orbital inclinations.

In contrast to rotating black holes, an intrinsic quadrupole component appears in the external fields of rapidly rotating neutron stars (NSs). For NSs with a stiff equation of state, this component can be several times larger than the Kerr quadrupole moment. The effect of a non-Kerr NS field induced by rapid NS rotation on the parameters of the marginally stable orbit and energy release in the equatorial boundary layer was analyzed by Sibgatullin and Sunyaev (1998, 2000a, 2000b) using an exact solution for a rotating configuration with a quadrupole moment (Manko *et al.* 1994) [approximate approaches with multipole expansions of the metric coefficients at large radii were developed by Laarakkers and Poisson (1998) and Shibata and Sasaki (1998)]. Our results refer to nonequatorial, nearly circular orbits in the fields of black holes and NSs with a soft equation of state at a moderate rotation frequency. Markovic (2000) considered finite-eccentricity orbits in a Kerr field and in the post-Newtonian approximation for a field with a finite quadrupole moment. The precession of orbits slightly inclined to the equatorial plane in the fields of rotating NSs was numerically calculated by Morsink and Stella (1999) and Stella *et al.* (1999).

Geodesics in the Kerr solution

It is well known that the equations of geodesics in Riemannian spaces = can be written in Hamiltonian form. The corresponding Hamilton–Jacobi equation is

$$Q \equiv (g^{i,j} S_{,i}\, S_{,j} + 1)/2 = 0, \qquad S_{,i} \equiv \frac{\partial S}{\partial x_i}, \tag{1}$$
$$i, j, ... = 0, 1, 2, 3.$$

The generalized momenta $p_i = S_{,i}$ are related to the 4-velocity components by the Hamilton equations

$$\frac{dx^j}{d\tau} = \frac{\partial Q}{\partial p_j} = g^{jk} p_k, \; \frac{dp_j}{d\tau} = -\frac{\partial Q}{\partial x^j} = -\frac{\partial g^{kl}}{\partial x^j} p_k p_l. \tag{2}$$

For the Kerr solution in Boyer–Lindquist coordinates, the contravariant metric components are

$$\begin{gathered} g^{00} = -\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2 c^2 \Delta}, \\ g^{\phi\phi} = \frac{(1 - 2rGM/c^2\rho^2)}{\Delta \sin\theta}, \\ g^{0\phi} = \frac{-2ar}{\rho^2 \Delta}, \quad g^{rr} = \frac{\Delta}{\rho^2}, \quad g^{\theta\theta} = \frac{1}{\rho^2}. \\ \Delta \equiv r^2 + a^2 - 2rGM/c^2; \quad \rho^2 \equiv r^2 + a^2 \cos^2\theta. \end{gathered} \tag{3}$$

[3]Below, s is called the inclination angle between the angular velocity vector of a compact object and the orbital surface for short.

The constant a is related to the angular momentum of a rotating black hole described by the Kerr solution by the equality $J = Mac = GM^2 j/c$, where j is the dimensionless Kerr parameter. Because of axial symmetry and stationarity, the Hamilton–Jacobi equation (1) for the Kerr metric has two cyclic coordinates, t and ϕ; therefore, Eq. (1) with the first integrals $p_t = -E = \text{const}$ and $p_\phi = L = \text{const}$ for the Kerr metric can be written as

$$\begin{gathered}(1-E^2)((r^2+a^2)^2 - \Delta a^2 \sin^2\theta) + \\ 4arLEGM/c^3 + (\rho^2 - 2GMr/c^2) \times \\ L^2/(c\sin\theta)^2 \Delta^2 S_{,r}^2 + \Delta S_{,\theta}^2 = 2r(r^2+a^2)GM/c^2.\end{gathered} \tag{4}$$

The existence of the complete integral of the Hamilton–Jacobi equation established by Carter (1968) is less obvious:

$$S = -Etc^2 + L\phi + \int \sqrt{\Theta(\theta)}d\theta + \int \sqrt{R(r)}\frac{dr}{\Delta},$$

$$\Theta(\theta) \equiv Q - \cos^2\theta(a^2c^2(1-E^2) + L^2/(\sin\theta)^2), \tag{5}$$

$$\begin{gathered}R(r) \equiv (cE(r^2+a^2) - La)^2 - \\ \Delta(r^2 + (L - aEc)^2 + Q.\end{gathered} \tag{6}$$

Instead of the constant Q, we introduce a constant $s < \pi/2$, which has the meaning of the minimum angle θ (turning point) reached on the bounded trajectory in question [in the notation of Wilkins (1972), the constant $s \equiv \theta_-$; see also Shakura (1987)]. According to (5), the constant Q can be expressed in terms of s:

$$Q = \cos^2 s(a^2c^2(1-E^2) + L^2/(\sin s)^2). \tag{7}$$

In that case, the angle θ on a bounded trajectory varies over the range $s < \theta < \pi - s$.

For stable trajectories on the $r = \text{const}$ surfaces, the following conditions must be satisfied:

$$R(r) = 0, \quad dR(r)/dr = 0 \tag{8}$$

(Bardeen *et al.* 1972; Wilkins 1972).

The newtonian analog of the Kerr solution

The newtonian analog of the supercritical Kerr solution and its potentials

In the Newtonian limit, $E^2 \approx 1 + 2H/c^2$ and $\Delta \approx r^2 + a^2$. Substituting these expressions in Eq. (4) yields an approximate Hamilton–Jacobi equation:

$$\begin{gathered}2H = \frac{L^2}{(r^2+a^2)\sin^2\theta} + \frac{r^2+a^2}{r^2+a^2\cos^2\theta}S_{,r}^2 + \\ \frac{1}{r^2+a^2\cos^2\theta}S_{,\theta}^2 + \frac{4arLGM/c}{(r^2+a^2)(r^2+a^2\cos^2\theta)} - \\ \frac{2GMr}{r^2+a^2\cos^2\theta} + \frac{4(GMra\sin\theta)^2}{(r^2+a^2\cos^2\theta)c^2(r^2+a^2)}.\end{gathered} \tag{9}$$

Note that Eq. (8) describes the trajectories of test particles in the field of a flat disk with radius a in special curvilinear coordinates r, θ, ϕ, which are related to Cartesian coordinates x, y, z by

$$r = \frac{1}{2}(\sqrt{x^2 + y^2 + (z - ia)^2} + \\ + \sqrt{x^2 + y^2 + (z + ia)^2}), \tag{10}$$

$$a \sin\theta = \frac{1}{2i}(\sqrt{x^2 + y^2 + (z + ia)^2} - \\ \sqrt{x^2 + y^2 + (z - ia)^2}), \quad \phi = \arctan y/x. \tag{11}$$

When passing from Cartesian coordinates x, y, z to orthogonal curvilinear coordinates r, θ, ϕ using formulas (10) and (11), the nonzero metric tensor components are expressed in terms of the curvilinear coordinates as

$$g_{rr} = \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}, \quad g_{\theta\theta} = r^2 + a^2\cos^2\theta, \\ g_{\phi\phi} = (r^2 + a^2)\sin^2\theta. \tag{12}$$

Hamiltonian (9) contains the contravariant metric tensor components (12), and it can be written in Cartesian coordinates as

$$H = \frac{(p + \mathbf{A}/c)^2}{2} - \Phi.$$

The corresponding Hamilton equations are

$$\frac{dp_k}{dt} = \Phi,_k - (p + \mathbf{A}/c)\frac{\partial \mathbf{A}/c}{\partial x^k}, \quad \frac{dx^k}{dt} = p_k + \mathbf{A}_k/c. \tag{13}$$

Eqs. (13) can be rewritten as the equations of motion in a gravitomagnetic field in the quasi-Newtonian approximation:

$$\frac{dv}{dt} = \nabla\Phi + \mathrm{curl}\mathbf{A} \times v/c, \tag{14}$$

where the Newtonian potential Φ and the gravitomagnetic vector potential $\mathbf{A}$ in a vacuum satisfy the Laplace equation. The equations of relative motion in a rotating coordinate system can be derived from Eqs. (14) if $\mathbf{\Omega} \times \mathbf{R}$, $\mathbf{R} = (x, y, z)$, is substituted for $\mathbf{A}$. In that case, $\mathrm{curl}\mathbf{A}/c = -2\mathbf{\Omega}$ and the second term on the right-hand side of (13) represents the Coriolis force.

For a flat Kerr disk, the Newtonian potential Φ is (Israel 1970; Zaripov *et al.* 1995)

$$\Phi = \frac{GM}{2\sqrt{x^2 + y^2 + (z + ia)^2}} + \\ \frac{GM}{2\sqrt{x^2 + y^2 + (z - ia)^2}}. \tag{15}$$

The vector $\mathbf{A}$ consists of the components $-g_{0\alpha}/g_{00}, \quad \alpha = 1, 2, 3$ (Landau and Lifshitz 1980). In the Newtonian limit in a vacuum, the vector of the gravitomagnetic field Ψ can be introduced instead of the vector potential $\mathbf{A}$:

$$\mathrm{curl}\mathbf{A} = 2\nabla\Psi. \tag{16}$$

The scalar Ψ in an axisymmetric case matches the imaginary part of the Ernst complex potential. For the special case of a Kerr disk, the scalar Ψ is (Zaripov *et al.* 1995):

$$\Psi = \frac{GM}{2i\sqrt{x^2+y^2+(z-ia)^2}} - \frac{GM}{2i\sqrt{x^2+y^2+(z+ia)^2}}. \tag{17}$$

Clearly, expressions (15) and (17) for Φ and Ψ satisfy the Laplace equation.

As follows from definition (10), the $r=\text{const}$ surface is neither a sphere nor an ellipsoid.

The precession of circular keplerian Orbits in the field of a gravitating rotating mass

For the Newtonian analog of a supercritical Kerr disk, $a > GM/c^2$. However, since $a < GM/c^2$ for black holes, in this subsection, we discard terms of the order of a^2 in expressions (15) and (17) for Φ and Ψ lest the order of accuracy of the model be exceeded. In that case, $\Phi \approx GM/r, \quad \Psi \approx -azGM/r^3$.

Let us introduce a coordinate system with the origin at the gravitating mass with the z axis directed along the spin axis. The equations of motion for free particles in this coordinate system are

$$\frac{du}{dt} = -\frac{GMx}{r^3} + \left(6a(yw-zv)\frac{z}{r^5} + 2v\frac{a}{r^3}\right)\frac{GM}{c}, \tag{18}$$

$$\frac{dv}{dt} = -\frac{GMy}{r^3} + \left(6a(zu-xw)\frac{z}{r^5} - 2u\frac{a}{r^3}\right)\frac{GM}{c}, \tag{19}$$

$$\frac{dw}{dt} = -\frac{GMz}{r^3} + \left(6a(xv-y=u)\frac{z}{r^5}\right)\frac{GM}{c}. \tag{20}$$

In our approximation, the complete integral of the Hamilton–Jacobi equation is [cf. formulas (5)–(6)]

$$S = -Ht + L\phi + \int \sqrt{\Theta(\theta)}d\theta + \int \sqrt{2R(r)}dr,$$

$$\Theta(\theta) \equiv L^2\cot^2 s - L^2\cot^2\theta. \tag{21}$$

$$R(r) \equiv H + \frac{GM}{r} - \frac{L^2}{2r^2\sin^2 s} - 2\frac{LaGM}{r^3c}. \tag{22}$$

In formulas (21) and (22), we chose the turning point for the angle θ as a constant: according to (20), θ varies over the range $s < \theta < \pi - s$, with $0 < s < \pi/2$.

For stable circular trajectories, the equalities $R(r) = 0$ and $dR(r)/dr = 0$ must be satisfied. Substituting expression (21) for a fixed radius r in these equations yields the corresponding energy H and angular momentum L:

$$H \approx -\frac{GM}{2r} - a\sqrt{(GMr)^3}\frac{\sin s}{cr^4}, \tag{23}$$

$$L \approx \sin s\sqrt{GMR} - 3aGM\frac{\sin^2 s}{cr}.$$

Consider the precession of circular orbits. From the Hamilton equations $\dot\theta = \partial H/\partial p_\theta$ and $= \dot\phi = \partial H/\partial L$, we have

$$\frac{d\theta}{dt} = \frac{L}{r^2}\sqrt{\cot^2 s - \cot^2\theta}, \tag{24}$$

$$\frac{d\phi}{dt} = \frac{L}{r^2 \sin^2\theta} + \frac{2aGM}{r^3 c}. \tag{25}$$

Integrating Eq. (24) yields

$$\begin{gathered}\phi - \phi_0 = \int \frac{Ldt}{r^2\sin^2\theta} + \frac{2aGMt}{r^3c} = \\ \int \frac{d\theta}{\sin^2\theta\sqrt{\mathrm{c}^2 s - \mathrm{c}^2\theta}} + \frac{2aGMt}{r^3c} = \\ \arcsin\left(\frac{\cot\theta}{\cot s}\right) + \frac{2aGMt}{r^3c}.\end{gathered} \tag{26}$$

When changing the variable in formula (26), we used Eq. (24).

Consider the case where all particle at $t = 0$ were in the $\sin\phi\sin\theta = \cos\theta\cot s$ plane or, introducing Cartesian coordinates, in the $y = z\cot s$ plane. According to equality (26), the particles will lie on the $y\cos(2aGMt/r^3c) - x\sin(2aGMt/r^3c) = z\cot s$ surface at time t. The closer the particles to the rotating gravitating center, the faster their precession. For a given radius r, the particle motion in an accretion disk can be represented as the rotation of a circumference inclined to the z spin axis at angle s with angular velocity $2aGM/r^3c$. In a coordinate system corotating with the circumference, the particle rotates with a Keplerian velocity. Thus, if viscous friction between adjacent orbits is disregarded, a tilted accretion disk cannot be in a steady state.

The energy and angular momentum of particles moving along geodesics orbits in the Kerr solution on the $r =$ const surfaces

In this section, for simplicity, we choose a system of units in which $c = G = M = 1$. To find stable orbits on the $r =$ const surfaces, we must solve the algebraic equations (8). The corresponding solutions in the $s = \pi/2$ equatorial plane were found by Ruffini and Wheeler (1970) for $j = 1$ and by Bardeen *et al.* (1972) for arbitrary j. For $s = 0$ and arbitrary j, the particle angular momentum L is zero. The corresponding expression for the energy was derived by Lightman *et al.* (1975). Inclined orbits at $j = 1$ were investigated by Wilkins (1972). In the case of charged rotating black holes the selected tilted bound geodesics were studied by Johnston and Ruffini in extreme case $j = 1$. In the case of charged rotating black holes the selected tilted bound geodesics were studied by Johnston and Ruffini in extreme case $j = 1$. Finally, Shakura (1987) derived formulas for the energy and angular momentum at arbitrary j and s by a complex, indirect method. Below, we show that derivation expressions for E and L from Eqs. (8) can be reduced to solving a quadratic equation and factorizing the numerator and denominator in the resulting fraction (to be subsequently canceled by a common factor).

Let us introduce new unknowns instead of E and L:

$$x_c \equiv \frac{E}{L}, \quad y_c = \frac{E^2 - 1}{L^2}. \tag{27}$$

Eqs. (8) can then be written as

$$y_c a_1 + x_c a_2 + (x_c^2 - y_c)(a_1 + a_3) + a_0 = 0, \tag{28}$$

$$y_c b_1 + x_c b_2 + (x_c^2 - y_c)(b_1 + b_3) + b_0 = 0, \tag{29}$$
$$b_i \equiv \frac{\partial a_i}{\partial r}|_{j,s}, \quad i = 0, 1, 2, 3.$$

Here,

$$a_0 = -(r^2 - 2r + \Delta \cos^2 s),$$
$$a_1 = r^4 + 2j^2 + j^2 r + j^2 \Delta \cos^2 s,$$

$$a_2 = -4jr, \quad a_3 = -p\Delta, \ p \equiv r^2 + j^2 \cos^2 s. \tag{30}$$

Eliminating y from Eqs. (28) and (29) yields the quadratic equation

$$x_c^2(a_1 b_3 - a_3 b_1) + x_c(a_2 b_3 - a_3 b_2) +$$
$$+ a_0 b_3 - a_3 b_0 = 0,$$

whose solution is given by

$$x_c = \frac{E}{L} = \frac{A + qB}{D \sin s}, \quad q \equiv \sqrt{r - j^2 \cos^2 s / r}. \tag{31}$$

In formula (31), we use the notation

$$A \equiv j \sin s(3r^4 - 4r^3 + j^2 r^2 \cos^2 s(j^2 r^2 - r^4)),$$
$$B \equiv rp\Delta, \tag{32}$$
$$D = -a_1 b_3 + a_3 b_1 = (r^2 - j^2 \cos^2 s) \times \tag{33}$$
$$(r^2 + j^2)^2 - 4j^2 r^3 \sin^2 s.$$

The numerator and denominator in (31) can be factorized as

$$A + qB = r((r^2 + j^2)q + 2jr \sin s) \times \tag{34}$$
$$(j \sin sq + r^2 + j^2 \cos^2 s - 2r),$$
$$D = r((r^2 + j^2)q + 2jr \sin s)((r^2 + j^2)q - 2jr \sin s). \tag{35}$$

Therefore, cancelling a common factor in the numerator and denominator in (31), we obtain

$$x_c = \frac{E}{L} = \frac{jq \sin s + r^2 + j^2 \cos^2 s - 2r}{(r^2 + j^2)q - 2jr \sin s) \sin s}. \tag{36}$$

Multiplying Eq. (28) by b_1 and Eq. (29) by a_1 and subtracting (29) from (28) yields

$$x_c^2 - y_c = \frac{1}{L^2} = \frac{x(a_1 b_2 - a_2 b_1) + a_1 b_0 - a_0 b_1}{D} \tag{37}$$
$$= \frac{A_1 + qB_1}{D \sin s},$$

$$A_1 \equiv 2jrp(2r^4 - 3r^3 + j^2r^2 + \cos^2 s j^2(r - j^2)),$$
$$B_1 \equiv p^2(r^3 - 3r^2 + j^2 + j^2 r).$$

The expression $A_1 + qB_1$ can also be factorized as

$$A_1 + qB_1 = p((r^2 + j^2)q + 2jr\sin s) \times \\ (2rjq\sin s + r^3 - 3r^2 + j^2\cos^2 s(r + 1)). \tag{38}$$

Hence,

$$L = \sqrt{\frac{D\sin s}{A_1 + qB_1}} = \\ \frac{((r^2 + j^2)q - 2jr\sin s)\sin s}{\sqrt{p(p - 3r + j^2\cos^2 s/r + 2jqr\sin s)}}. \tag{39}$$

Using formula (36) for E/L, we obtain with (39)

$$E = \frac{p - 2r + jq\sin s}{\sqrt{p(p - 3r + j^2\cos^2 s/r + 2jqr\sin s)}}. \tag{40}$$

Formulas (39) and (40) are the sought-for expressions for the particle energy and angular momentum in inclined circular orbits.

We determine the coordinate radius of the marginally stable orbit, r_*, from the condition of E given by (40) being at a minimum. Representing r_* as a Taylor expansion in terms of j, we obtain r_* as a function of j, s:

$$r_* \approx 6 - 4\sqrt{\frac{2}{3}} j\sin s + .. \approx 6 - 3.266 j\sin s - \\ j^2(0.5 + 0.1111\cos 2s) + j^3(-0.2532\sin s - \\ 0.0567\sin 3s) + j^4(-0.1196 + 0.0206\cos 2s + \\ 0.0162\cos 4s) + j^5(-0.11\sin s - 0.02\sin 3s + \\ 0.0025\sin 5s)\ldots \tag{41}$$

At $s = 0$, the first expansion terms for r_* are

$$r_* \approx 6 - 0.6111j^2 - 0.0828j^4 - 0.211j^6 - \\ 0.0066j^8 - 0.0023j^{10} + \ldots$$

Substituting the Taylor expansion (41) in (39) and (40) yields Taylor expansions for the binding energy and angular momentum in the marginally stable orbit:

$$1 - E_* \approx 0.0572 + 0.0321j\sin s + j^2(0.0131 - \\ 0.0087\cos 2s) + j^3(0.0143\sin s - 0.002\sin 3s) + \\ j^4(0.0065 - 0.006\cos 2s + 0.0003\cos 4s) + \\ j^5(0.0087\sin s - 0.002\sin 3s)\ldots,$$

$$\frac{L_*}{\sin s} \approx 3.4641 - 0.9428j\sin s - j^2(0.1123 + \\ 0.1443\sin^2 s) - j^3(0.1178\sin s + 0.0175\sin^3 s) + \\ j^4(-0.0138 - 0.0878\sin^2 s + 0.0138\sin^4 s) + \\ j^5(-0.0327\sin s + 0.0467\sin^3 s + 0.016\sin^5 s)\ldots \tag{42}$$

We supplement expansions (41) and (42) with the values of the corresponding functions at $j = 1$ numerically constructed by least squares for an arbitrary fixed orbital inclination to the black-hole spin axis:

$$
\begin{gathered}
1 - E_* \approx 0.4222 - 0.8314 \cos^2 s + 0.264 \cos^4 s + \\
0.8605 \cos^6 s - 0.656 \cos^8 s, \\
r_* \approx 1 + 6.2005 \cos^4 s - \\
7.1816 \cos^8 s + 5.1365 \cos^{12} s,
\end{gathered}
$$

$$
\begin{gathered}
L_* \approx \sin s \left(1.1876 + 2.5913 \cos^2 s - \right. \\
\left. 0.9948 \cos^4 s + 0.494 \cos^6 s\right).
\end{gathered}
\tag{43}
$$

Note that for $s = 0$ and $j = 1$, the radius of the marginally stable orbit is $1 + \sqrt{3} + \sqrt{3 + 2\sqrt{3}} \approx 5.2745$ (in units of GM/c^2).

Periastron and nodal precession of orbits on the $r =$ const surfaces

Choose the angle θ as a parameter on a bounded trajectory. It then follows from (5)–(7) that

$$
\begin{gathered}
\frac{d\phi}{d\theta} = \frac{(2jEr + L(\Delta/\sin^2\theta - j^2))}{\Delta\sqrt{\Theta}}, \\
\Theta \equiv (1 - \frac{\sin^2 s}{\sin^2\theta}) \left(\sin^2\theta j^2 (1 - E^2) + \frac{L^2}{\sin^2 s}\right),
\end{gathered}
$$

$$
\frac{dt}{d\theta} = \frac{-2jrL + E(r^2 + j^2\cos^2\theta)\Delta + 2rE(r^2 + j^2))}{\Delta\sqrt{\Theta}}. \tag{44}
$$

Expressions (39) and (40) for the energy and angular momentum in circular orbits must be substituted in these formulas for E and L.

Integrating the right-hand parts of expressions (44) over θ from s to $\pi/2$ yields an expression for a quarter of the variation in azimuthal angle $\Delta\phi$ and for a quarter of the period T in which the particle runs from a minimum latitudinal angle s to its maximum $\pi - s$ and back. The integration result can be expressed in terms of the elliptic functions that contain E, L, and s as parameters [instead of the constant Q, we inserted the constant s into formula (7), which has the meaning of orbital inclination to the black-hole spin axis). Using the notation for the complete elliptic integrals of the first, $K(x)$, second, $E(x)$, and third, $\Pi(n, x)$, kinds, we have

$$
\begin{gathered}
T = 4A \left(K(k)(jL - j^2E + (Er^2 + Ej^2 - jL)\times \right. \\
\left. \frac{(r^2 + j^2)}{\Delta}) + (K(k) - E(k))\frac{E}{A^2(1 - E^2)}\right); \\
\Delta\phi = 4A(L\Pi(-\cos^2 s, k) + (2rEj - j^2L)K(k)),
\end{gathered}
$$

$$A \equiv (L^2/\sin^2 s + j^2(1-E^2))^{-1/2}; \qquad (45)$$
$$k^2 \equiv cos^2 s j^2 (1-E^2) A^2.$$

In contrast to Johnston and Ruffini (1974), who first wrote corresponding formulas in form of elliptic integrales for any j, in formulas (45), we inserted explicit expressions for all quantities in terms of the turning-point angle $\theta_- \equiv s$ in the orbit. Expressions (39) and (40) must be substituted for E and L, respectively.

For low-eccentricity orbits[4]($\delr = \epsilon \sin\xi, \quad \epsilon << r$), it follows from the Hamilton–Jacobi equation (4) that

$$\frac{d\xi}{d\theta} = \frac{\sqrt{-R''/2}}{\sqrt{\Theta}}, \quad R'' \equiv \frac{d^2R}{dr^2} = \qquad (46)$$
$$-2\left(6(1-E^2)r^2 - 6r + (1-E^2)j^2\sin s^2 + \frac{L^2}{\sin^2 s}\right).$$

From Eq. (46) for the periastron rotation frequency ν_r, we have

$$\omega_r = 2\pi\nu_r = \frac{\Delta\xi}{T} = 4A\frac{\sqrt{R''/2}}{T}K(k). \qquad (47)$$

Note that the frequencies $\Delta\phi/2\pi T$ and $1/T$ are commonly denoted by ν_ϕ and ν_θ, respectively, so the sought-for nodal precession frequency is $\nu_{\rm nod} = \nu_\phi - \nu_\theta$. The periastron precession frequency of an orbit is $\nu_{\rm per} = \nu_\phi - \nu_r$ (Merloni *et al.* 1999).

To clearly present the result, let us derive asymptotic formulas from (45) and (47) in the form of Taylor expansions in powers of j by taking into account the dependence of the first integrals E, L, and Q on j and s given by formulas (39), (40), and (7).

The constants E and $\tilde{L}$ enter into the formulas for $\Delta\phi$ and T via the ratios $E/\tilde{L}$ and $(1-E^2)/L^2, Q/L^2$. Let us write out the series expansions of these ratios with the accuracy that will be required to calculate the nodal precession frequency $(\Delta\phi/2\pi - 1)/T$ up to terms of the order of j^3 inclusive:

$$\frac{E}{L} = \frac{r-2}{r^{3/2}\sin s} + j\frac{3r-4}{r^3} +$$
$$j^2\frac{(r + 0.5r^2 + \sin^2 s(8 + 7r - 1.5r^2))}{r^{9/2}\sin s} + \ldots,$$
$$\frac{1-E^2}{L^2} = \frac{r-4}{r^3\sin^2 s} + j\frac{8(r-2)}{r^{9/2}\sin s} + \ldots$$

$$\frac{Q}{L^2} = \cot^2 s(1 + j^2\frac{(r-4)\sin s}{2r^3} - \qquad (48)$$
$$j^3\frac{4(r-2)\sin s}{r^{9/2}} + \ldots).$$

[4]Syer and Clarke (1992) considered the possible existence of stationary disks in the equatorial plane in which the particle orbits were constant-eccentricity ellipses. Clearly, allowance for general-relativity effects will result in the intersection of orbital trajectories in such disks, because the orbital periastron precesses. However, in the Newtonian theory, unstable disturbance modes with a radially variable eccentricity disturbance exist in such disks (Lyubarskij *et al.* 1994). Arbitrary accretion-disk disturbances apparently produce spiral waves in the disk structure (Spruit 1987).

Using (48), we obtain from expressions (45) and (47) for $\Delta\phi$, T, and $\Delta\xi$

$$\frac{\Delta\phi}{2\pi} - 1 = \frac{2j}{r^{3/2}} - \frac{j^2}{2r^3}3(r-4)\sin s + \frac{j^3}{r^{9/2}}(2 - 1.5r + (18 - 7.5r)\sin^2 s) + \dots$$
$$\frac{T}{2\pi} = r^{3/2} + 3j\sin s + \frac{j^2}{4r^{3/2}} \times (8 + 3r + (24 - 9r)\sin^2 s) + \dots$$
$$\left(\frac{\Delta\xi}{2\pi}\right)^2 = \frac{r-6}{r} + \frac{12j}{r^{5/2}}(r-2)\sin s \frac{j^2}{r^4} \times \left(\frac{3}{2}r^2 + 15r - 12 - \sin^2 s(r-4)(\frac{15}{2}r - 21)\right) + \dots \tag{49}$$

In the limit $s \to 0$, when the spin axis becomes parallel to the tangential plane to the accretion-disk surface, the following formulas hold:

$$\frac{\Delta\phi}{2\pi} - 1 = \frac{2j}{r^{3/2}} + \frac{j^3}{r^{9/2}}(2 - 1.5r) + \frac{3j^5}{r^{15/2}}\left(\frac{3}{2} - \frac{7}{4}r + \frac{19}{32}r^2\right) + \dots$$
$$\frac{T}{2\pi} = r^{3/2} + \frac{j^2}{4r^{3/2}}(8 + 3r) + + \frac{3j^4}{64r^{9/2}}(7r^2 - 48r + 80) + \dots$$
$$\left(\frac{\Delta\xi}{2\pi}\right)^2 = \frac{r-6}{r} + \frac{3j^2}{2r^4}\left(r^2 + 10r - 8\right) + \frac{3j^4}{32r^7}(-352 + 576r - 286r^2 + 9r^3)\Big) + \dots \tag{50}$$

From formulas (49) for the Keplerian frequency ν_ϕ, the nodal precession frequency $\nu_{\rm nod}$, and the periastron rotation frequency ν_r of a low-eccentricity orbit around the black-hole spin direction, we derive the compact formulas

$$\omega_\phi = 2\pi\nu_\phi = \frac{\Delta\phi}{T} = \frac{1}{r^{3/2}}\left(1 + \frac{(2 - 3\sin s)j}{r^{3/2}} + \frac{((12+9r)\sin^2 s - 8 - 9r)j^2}{4r^6} + \dots\right);$$
$$\omega_{nod} = 2\pi\nu_{nod} = (\Delta\phi - 2\pi)/T \approx \frac{2j}{r^3} - j^2\frac{1.5\sin s}{r^{7/2}} + j^3\frac{-2 - 3r + (6 + 1.5r)\sin^2 s}{r^6} + \dots$$

$$(\omega_r)^2 = (2\pi\nu_r)^2 = \frac{r-6}{r^4} + \frac{6j}{r^{11/2}}\sin s(r+2) + \frac{j^2}{r^7}(12 + 20r - 3\sin^2 s(1+r)(10+r)) + \dots \tag{51}$$

The first term in the formula for nodal precession was found by Lense and Thirring (1918) (see also Wilkins 1972). In a coordinate system rotating with the frequency $\nu_{\rm nod}$, the Keplerian orbit is stationary.

In the limit $s \to 0$, when the black-hole spin axis becomes tangential to the disk surface, we can derive the following expansions for the nodal precession frequency, the periastron rotation frequency, and the Keplerian latitudinal frequency from formulas (45) and (47) using (50)[5]

$$2\pi\nu_{\rm nod} = (\Delta\phi - 2\pi)/T \approx \frac{2j}{r^3}\left(1 - j^2\frac{3r+2}{2r^3} + j^4\frac{8+54r+27r^2}{16r^6} + \ldots\right)$$

$$(2\pi\nu_r)^2 = \frac{r-6}{r^4} + \frac{4j^2}{r^7} - \frac{3j^4}{r^{10}}(r^3+128r^2+172r+32) + \ldots$$

$$2\pi\nu_\theta = \frac{1}{r^{3/2}} - \frac{j^2}{4r^{9/2}}(3r+8) + \frac{j^4}{64r^{15/2}}(16+336r+15r^2) + \ldots \qquad (52)$$

To compare formulas (51) and (52) with observations, we should eliminate the radius from them and express the frequencies $\nu_{\rm nod}$ and ν_r as functions of the Keplerian frequency ν_ϕ. The functions $\nu_{\rm nod}(\nu_\phi)$ and $,\nu_r(\nu_\phi)$ may be said to be given by (51) and (52) in parametric form. An increase in ν_2 and a decrease in the frequency difference between the two peaks $\nu_2 - \nu_1$ during observations implies the transition of a radiating clump to an orbit closer to the black hole and the approach of its orbit to the marginally stable orbit. When observations are accumulated, formulas (51) and (52) make it possible to determine the orbital inclination to the spin axis of a slowly rotating black hole or neutron star. When analyzing the energy release in the boundary layer and in an extended disk for a rapidly rotating neutron star, we should took into account the appearance of an intrinsic quadrupole moment that exceeds the Kerr one (see Sibgatullin and Sunyaev 1998, 2000a, 2000b). We emphasize that formulas (51) and (52) are not related to the weak-field approximation and are valid up to the marginally stable orbit.

If the disk almost lies in the equatorial plane of a black hole ($s = \pi/2$), then formulas (51) give Taylor expansions of the formulas by Okazaki *et al.* (1987) and Kato (1990) (see also Merloni *et al.* 1999):

$$\omega_\phi = \frac{1}{r^{3/2}+j},$$

$$\omega_{nod} = \frac{1-\sqrt{1-4j/r^{3/2}+3j^2/r^2}}{r^{3/2}+j},$$

$$\omega_r^2 = \frac{1-6/r+8j/r^{3/2}-3j^2/r^2}{(r^{3/2}+j)^2}. \qquad (53)$$

[5]Caution must be exercised when passing to the limit $s \to 0$; for example, when $s \to 0$, $L/(\sqrt{\Theta}\sin^2\theta) \to \delta(\sin\theta)$, where $\delta(x)$ denotes the Dirac δ function.

The nodal frequency of the marginally stable orbit for an arbitrary orbital inclination angle s

The following relations (Sibgatullin and Sunyaev 1998, 2000a) hold for the marginally stable orbit in the equatorial plane in a Kerr field, in which the reciprocal radius of the marginally stable orbit acts as a parameter:

$$j = \frac{4\sqrt{x} - \sqrt{3-2x}}{3x}, \quad E = \sqrt{\frac{3-2x}{3}},$$
$$L = \frac{2}{3\sqrt{3x}}(2\sqrt{x}\sqrt{3-2x} + x), \quad x \equiv 1/r.$$

Let us substitute these expressions in (53):

$$\omega_\phi = \frac{3x^{3/2}}{3 - \sqrt{x}\sqrt{3-2x} + 4x},$$
$$\omega_{nod} = \omega_\phi(1 + \sqrt{\frac{2}{3}}(\sqrt{x} - \sqrt{3-2x})), \quad \nu_r = 0. \tag{53a}$$

If the parameter x is eliminated from (53a), then we derive the following dependence by least squares:

$$\omega_{nod} \approx 0.1872(\omega_\phi - \sqrt{6}/36) +$$
$$1.7246(\omega_\phi - \sqrt{6}/36)^2 + 1.2064(\omega_\phi - \sqrt{6}/36)^3. \tag{53b}$$

We emphasize that formula (53b) accurately describes the limiting dependence of the nodal precession frequency on Keplerian frequency at $s = \pi/2$ in the marginally stable orbit over the entire ω_ϕ range from 1/26 to 0.5, which corresponds to the range of Kerr parameter from -1 to 1. To derive dimensional dependences, we must make the following substitutions in all formulas: $23.1\omega_{\rm nod} = \nu_{\rm nod}(M/1.4M_\odot), 23.1\omega_\phi = \nu_\phi(M/1.4M_\odot), \ldots$, where frequencies $\nu_\phi, \nu_{\rm nod}$ will be given hereafter in kHz.

To determine the nodal precession frequency of the marginally stable orbit for an arbitrary orbital inclination s, we use formulas (45) in which for E, L, r, we substitute their expressions in the marginally stable orbit as functions of j and s, (41) and (42). Below, we give the highly accurate approximation dependences for the nodal precession frequency and the Keplerian frequency of the marginally stable orbit, in kHz, on Kerr parameter in the entire j range ($-1 \le j \le 1$) that we derived by analyzing our numerical calculations for various inclinations of this orbit to the black-hole spin axis:

$$
\begin{aligned}
s=0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + 0.0448j^3 + \\
& 0.0308j^5 + 0.007j^7), \\
& \nu_\phi \approx \frac{1.4M_\odot}{M}(1.5716 + 0.1933j + 0.1539j^2 + \\
& 0.0712j^3 + 0.0852j^4 + 0.0533j^5); \\
s=10^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + 0.0431j^2 + \\
& 0.0674j^3 + 0.054j^4 + 0.0004j^5 - 0.0414j^6 + \\
& 0.038j^7 + 0.0432j^8), \\
& \nu_\phi \approx \frac{1.4M_\odot}{M}(1.5716 + 0.3875j + 0.2206j^2 + \\
& 0.0675j^3 + 0.1681j^4 + 0.1289j^5); \\
s=30^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.1394j^2}{1-0.82j}), \\
s=45^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.1923j^2}{1-0.87j}), \\
s=60^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.2499j^2}{1-0.92j}), \\
s=80^0: \quad & \nu^*_{nod} \approx \frac{1.4M_\odot}{M}(0.2139j + \frac{0.2535j^2}{1-0.976j}).
\end{aligned}
\tag{54}
$$

Note that, in contrast to the preceding formulas, the last two formulas in (54) approximate the numerical data in the range $-1 \le j \le 0.99$.

The corresponding limiting dependence of the nodal precession frequency on Keplerian frequency for $s = \pi/6, \pi/4, \pi/3, 4\pi/9$ can be derived from numerical calculations by least squares (here, $\tilde{\omega_\phi} \equiv \omega_\phi - \sqrt{6}/36$):

$$
\begin{aligned}
s=30^0: \quad & \omega_{nod} \approx 0.3189\tilde{\omega_\phi} + 0.5205(\tilde{\omega_\phi})^2 + \\
& 4.37(\tilde{\omega_\phi})^3, \quad 0.049 \le \omega_\phi \le 0.171; \\
s=45^0: \quad & \omega_{nod} \approx 0.2416\tilde{\omega_\phi} + 1.5953(\tilde{\omega_\phi})^2 - \\
& 0.2723(\tilde{\omega_\phi})^3, \quad 0.044 \le \omega_\phi \le 0.257; \\
s=60^0: \quad & \omega_{nod} \approx 0.2031\tilde{\omega_\phi} + 1.7744(\tilde{\omega_\phi})^2 + \\
& 0.5149(\tilde{\omega_\phi})^3, \quad 0.041 \le \omega_\phi \le 0.414; \\
s=80^0: \quad & \omega_{nod} \approx 0.1846\tilde{\omega_\phi} + 1.7929(\tilde{\omega_\phi})^2 + \\
& 1.0862(\tilde{\omega_\phi})^3, \quad 0.039 \le \omega_\phi \le= 0.485.
\end{aligned}
\tag{54a}
$$

In formulas (54a), we give the ranges of Keplerian frequencies that correspond to the range of Kerr parameter from -1 to $+1$. Consider an example. Let the black-hole mass be $2.2M_\odot$ and the Keplerian rotation frequency be 1.2 kHz. Since the rotation frequency in the marginally stable Keplerian orbit of a nonrotating black hole with $M = 2.2M_\odot$ is $23.1 \times 1.4/2.2 \times \sqrt{6}/36 \approx 1$ kHz, $\tilde{\omega_\phi} = 0.2 \times 2.2/(1.4 \times 23.1) \approx 0.0136$. According to (54a), the nodal precession frequency of the marginally stable orbit is: $\nu_{\rm nod} \approx 42$ Hz for $s = \pi/2$; $\nu_{\rm nod} \approx 52.6$ Hz for $s = \pi/4$; $\nu_{\rm nod} \approx 65.3$ Hz for $s = \pi/6$; $\nu_{\rm nod} \approx 93.6$ Hz for $s = \pi/18$; and

$\nu_{\rm nod} \approx 123$ Hz for $s \to 0$. We see that the nodal precession frequency significantly depends on the orbital inclination and changes by a factor of 3 as the inclination changes! If the nodal precession frequency is identified with the horizontal-branch oscillation frequency $\nu_{\rm HBO}$, then it becomes possible to determine the inclination of the marginally stable orbit from the known black-hole mass, the Keplerian frequency, and the nodal precession frequency. At $j << 1$, the nodal precession frequency of the marginally stable orbit is related to the Keplerian frequency by

$$\omega_{nod} \approx \frac{2}{2 + 9\sin s}\tilde{\omega_\phi}. \tag{54b}$$

At a fixed inclination s, the precession frequency ν at the marginally stable orbit monotonically increases with j and reaches a maximum at $j = 1$. For the mass $M = 1.4M_\odot$ and $s = 0$, it is 296 Hz.

The larger the inclination s at fixed j, the larger the precession frequency $\nu_{\rm nod}$. As an illustration, let us write out the approximation formula for $\nu_{\rm nod}$ at $j = 0.998$ [Thorne (1974)][6]:

$$\nu_{nod} \approx 0.29 + 0.7783\sin s + \frac{17.1094\sin^6 s}{1 + 1.35\sin^4 s} \text{in kHz}. \tag{55}$$

Conclusions

According to the ideas explicitly formulated by Bardeen and Petterson (1975) and Bardeen (1977), a geometrically tilted accretion disk is most commonly modeled as a set of rings with the center at the coordinate origin that smoothly turn with changing radius under the effect of viscous torques and gravitomagnetic forces. This approach underlies the existing theories of tilted accretion disks [Papaloizou and Pringle (1983) and Markovic and Lamb (1998, 2000) for small disk deviations from the equatorial plane; Pringle (1992, 1997) for finite disk deviations from the equatorial plane), which are based on a vector equation for the conservation of angular momentum. Some ideas developed by Shakura and Sunyaev (1973, 1976) for flat disks can be used to study the physics of tilted accretion disks. Van Kerkwijk *et al.* (1998) explained the puzzling spindown and spinup of some X-ray pulsars by the fact that the accretion-disk tilt in the inner regions could become larger than 90 degrees!

The effects considered above take place for the inner nonstationary part of the disk, where viscous torques may be disregarded. Matter is not accumulated in the marginally stable orbit because of its instability. In contrast to the statement by Stella *et al.* (1999) that the orbital inclination affects weakly the nodal precession frequency, our detailed analysis shows that this effect is significant. There are probably preferential inclinations of the marginally stable orbits, and there is no need to introduce even harmonics of the nodal frequency to reconcile observations with theory.

[6]Studying the evolution of a black hole under the effect of disk accretion in the equatorial plane, Thorne adduced arguments that the Kerr parameter cannot be larger than 0.998 in this case when the radiation from inner disk regions is taken into account. Since the black hole mostly captures photons with a negative angular momentum (opposite to the black-hole rotation), the black-hole spinup by disk accretion is counteracted by its spindown via the predominant capture of the photons propagating against the rotation.

Acknowledgments

I wish to thank Prof. Sunyaev, who drew my attention to the problem considered and for fruitful discussions.

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Translated by V. Astakhov